Journal of Computational Physics **154,** 237–241 (1999)
Article ID jcph.1999.6308, available online at http://www.idealibrary.com on IDEAL®

# NOTE

## Good Neighborhoods for Multidimensional Van Leer Limiting

Blair Swartz

*Group T-7, MS B284, University of California, Los Alamos National Laboratory, Los Alamos, New Mexico 87545*
E-mail: bks@lanl.gov

Received March 3, 1999

*Key Words:* Van Leer limiting; finite volume; flux corrected transport.

Van Leer limiting uses nearby cell-means of a function (integral mean-values—weighted by a prescribed positive density—that are taken over each of a collection of nearby computational cells) to restrict the range of values allowed to a linear approximation of the function on a given central cell. These nearby cells—whose cell-means are actually involved in the limiting—are called the central cell's *neighbors*; and the set of these neighbors is called the central cell's *neighborhood*. The use of certain neighborhoods in multidimensional Van Leer limiting can force even linear functions to be subject to restriction over the central cell. A simple geometric property characterizes those neighborhoods whose use would not require that any linear functions be limited. (Such a neighborhood is called a *good neighborhood for Van Leer limiting* since its use would not preclude second-order accuracy in the local linear approximation of a smooth function by one that is Van Leer limited—unless the additional, here unspecified, details for obtaining the approximation preclude it by themselves.) The characterization is as follows, where it is presumed that the cells lie in a finite dimensional vector space: *One has chosen a good neighborhood for a given central cell if and only if the convex hull of the centroids of its associated neighbors contains that central cell*.

Details now follow.

In the context of Van Leer limiting when locally approximating a function, all cell-means are prescribed. The approximating functions are to be piecewise-linear with a possibly different linear function on each mesh cell. Dukowicz and Kodis [2, pp. 213–214, Sect. 2.3.1] describe a form of Van Leer limiting on these approximations that is applicable in one or more dimensions as follows:

DEFINITION. Multidimensional Van Leer limiting. On all cells the mean value is to remain as prescribed—but, on a given central cell, one is restricted to those approximating



0021-9991/99

linear functions whose *range of values lies in the smallest interval containing the given cell-means associated with the central cell's neighboring cells.*

(Should the cell's own mean value not lie in the latter range, one chooses a gradient of zero. Surprisingly, this can also preclude second-order accuracy when approximating smooth functions using an inappropriate set of neighboring cells in more than one dimension. This note's penultimate paragraph contains an example.) A definition similar to Dukowicz and Kodis' is found in Barth and Jespersen [1, p. 5]. In this note the cell centroids and the cell-means are to be based on the same (possibly constant) prescribed positive density function.

Such limiting in one space dimension (Van Leer [5, pp. 289–290; 6, pp. 117–118]) does not affect any globally defined linear function as long as it involves both a left neighbor and a right neighbor. That is to say, *any* given linear function $L$ satisfies the Definition's criteria when it both specifies the mean cell values on the neighboring mesh intervals and is simultaneously used as the local linear candidate function on the central interval. This is independent of which interval(s), out of a collection of possibly irregular mesh intervals to the left, that one takes as left-neighbor(s); and which interval(s) to the right, as right-neighbor(s). (Consequently, Van Leer limiting does not, by itself, preclude second-order accuracy on any such mesh in one dimension.) This one-dimensional result is also a consequence of the following multidimensional.

*Remark.* There exists no linear function that will be subject to Van Leer limiting on a given central cell if and only if that cell is contained in the convex hull of the centroids of its neighbors. Hence, one has chosen a good neighborhood for a given central cell if and only if the convex hull of the centroids of its associated neighbors contains that central cell.

Examples of good and not-so-good neighborhoods (constant density) are illustrated in Fig. 1. The convex hull appropriate to Fig. 1a is enclosed by the dotted contour. Barth and Jespersen [1, p. 6] reject two gradient estimates involving configurations like Fig. 1e—but not because of the possibility that the limiting *itself* can preclude exact results when approximating linear functions. They accept configurations like Fig. 1g for their estimates.

Inspired by Fig. 1f, suppose the interior points $x$ of a tetrahedron $C$ satisfy $L_i(x) < 0$, $1 \le i \le 4$ for some four linear functions $L_i$. Then any neighborhood of $C$ that consists of four cells $(C_i)_1^4$ having centroids $(\bar{c}_i)_1^4$ that satisfy $(L_j(\bar{c}_i) > 0,\ j \ne i)_{i=1}^4$ would be a good neighborhood of $C$. Geometrically, this asks that each of the tetrahedron's four "exterior vertex solid angles" contain a neighboring cell's centroid.

*Proof of the remark.* Suppose $x_0$ lies in the given cell $C$ but outside the convex hull $\mathcal{H}_{\bar{\mathcal{N}}(C)}$ of the centroids $\bar{\mathcal{N}}(C)$ of $C$'s chosen "neighboring cells" $\mathcal{N}(C)$. Then a linear function that is subject to Van Leer limiting can be constructed as follows. Let $P$ be a (hyper) plane that (strictly) separates $x_0$ from $\mathcal{H}_{\bar{\mathcal{N}}(C)}$, and let $\nu$ be the unit normal for $P$ that points from the convex hull towards $x_0$. The linear function

$$L(x) := \nu \cdot (x - x_0)$$

increases in the direction of $\nu$ and is negative on $\mathcal{H}_{\bar{\mathcal{N}}(C)}$ (in particular, at the centroids $\bar{\mathcal{N}}(C)$ of $C$'s "neighbors"), and the part of $C$ that lies in the positive half-space for $L$ is nonempty.

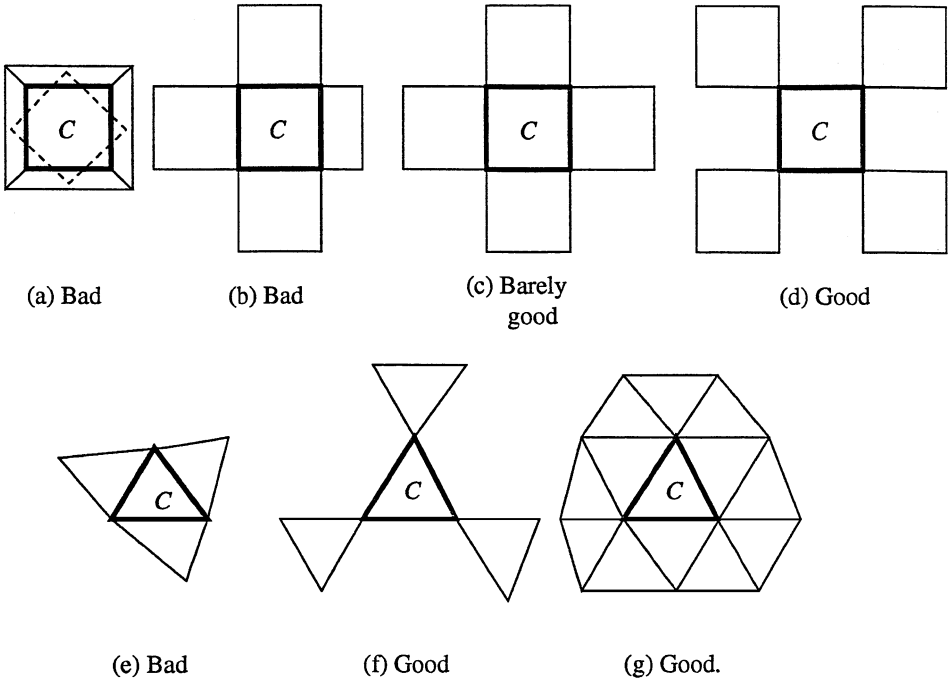


**FIG. 1.** Some good and some not-so-good neighborhoods of a central cell $C$.

A constant $c > 0$ can be added to $L$,

$$L_c := L + c,$$

so that $L_c$ is positive both on $C$ and on $\mathcal{H}_{\bar{\mathcal{N}}(C)}$. The mean value of $L_c$ over one of $C$'s neighboring cells is the value of $L_c$ at that neighbor's centroid. Since $L$ was negative at all of these neighboring cells' centroids but was 0 at $x_0$, $L_c$'s maximum over $C$ is at least $c$, which exceeds the maximum of $L_c$'s cell-means over the neighborhood. So, $L_c$ is subject to Van Leer limiting.

On the other hand, suppose $C$ is contained in the convex hull $\mathcal{H}_{\bar{\mathcal{N}}(C)}$, a convex polyhedron whose vertices are a subset of $\bar{\mathcal{N}}(C)$. Then a given linear function $L$ takes on its extreme values over $\mathcal{H}_{\bar{\mathcal{N}}(C)}$ at points in $\bar{\mathcal{N}}(C)$. So $L$'s values on $C$ lie between these extreme values, since $C$ is contained in $\mathcal{H}_{\bar{\mathcal{N}}(C)}$. These extreme values are also the extremes of $L$'s cell-means over $\bar{\mathcal{N}}(C)$. Thus, $L$ is not subject to Van Leer limiting on $C$. This completes the proof of the Remark.

*Some Properties of Good Neighborhoods*

1. In typical irregular grids in more than one dimension, there always exists a good neighborhood for a grid-cell that is sufficiently far from grid boundaries.

2. Typical choices of a minimal number of immediate neighbors may not yield a good neighborhood; see, e.g., Fig. 1b and Fig. 1e.

3. A set of cells that contains a good neighborhood as a subset is itself a good neighborhood; e.g., compare Fig. 1f with Fig. 1g.

4. Cells in good neighborhoods may overlap (but they don't in most applications).

5. Being a good neighborhood is invariant under linear maps of the mesh.

6. The Van Leer limited linear approximations of a given linear function are subject to the same restrictions—if any—on their gradients when the mesh is uniformly expanded or shrunk by a change in scale.

(If the density is not constant, then the last two are true only asymptotically—as the mesh gets small—and where the density is smooth.)

Not being allowed to reproduce linear functions can affect convergence rates of local linear approximations. For the following discussion of convergence, suppose (a) that the density is constant; (b) that the mesh is not too distorted: the ratio of the size of a cell's circumscribing sphere to its inscribing sphere, taken over all cells, is bounded; (c) that—absent Van Leer limiting—the local approximation algorithm is exact for linear functions and $O(h^2)$ accurate for smooth nonlinear functions (with $h$ being the diameter of local mesh cells); and (d) that the gradient of the function being approximated is not zero nearby.

*Consequences for Convergence*

1. The use of bad neighborhoods reduces the potential accuracy of Van Leer limited approximations of certain smooth functions to first-order. For example, if the linear function $L$ is subject to Van Leer limiting on some mesh cell, then, as the grid is shrunk, none of the (limited) approximating linear functions has a gradient converging locally to grad $L$, and the (limited) linear approximations themselves converge locally to $L$ with no better than first-order $O(h) \neq o(h)$ accuracy. A quadratic function with the same linear part $L$ fares no better asymptotically.

2. For sufficiently good neighborhoods, however, local linear approximations of smooth (e.g., twice differentiable) functions are not subject to Van Leer limiting as $h$ gets small and, so, converge with second-order $O(h^2)$ accuracy. "Sufficiently good," here, means that the shadow (via orthogonal projection) of the central cell $C$ on any line is well inside the shadow of the convex hull $\mathcal{H}_{\bar{\mathcal{N}}(C)}$ on that same line. By "well inside" is meant that (a) it is inside, and (b) it is more than $o(h)$ from the either end of the shadow of $\mathcal{H}_{\bar{\mathcal{N}}(C)}$. For this it is also supposed that the approximation algorithm, absent Van Leer limiting, is stable against perturbations as $h$ gets small—i.e., changes in the data of order $e$ in size produce changes in the (unlimited) approximation that are of the same order $O(e)$ in size.

The author believes that, in these circumstances and as $h$ goes to zero, Van Leer limiting takes effect *only* if the function being approximated is too rough to be entitled to the $O(h^2)$ accuracy one associates with local approximation by linear functions.

In current computational practice the set of points $\mathcal{C}$ over which the values of a linear function $L$ are being limited, and the set of points $\bar{\mathcal{N}}$ which are associated with the values involved in the limiting, may differ from the sets $C$ and $\bar{\mathcal{N}}(C)$, respectively, used above. In this more general context, it follows as before that *$\bar{\mathcal{N}}$ is a "good set of points" for the Van Leer limiting of linear functions over $\mathcal{C}$ if and only if $\mathcal{C}$ is a subset of $\mathcal{H}(\bar{\mathcal{N}})$, the convex hull of $\bar{\mathcal{N}}$.*

For example, Durlofsky *et al.* [3, pp. 66–67] consider four triangles grouped as in Fig. 1e. But they limit the values of a linear function over a set $\mathcal{C}$ of only three points from the central triangle $C$, namely, at the midpoints of $C$'s three sides. The values doing the limiting in their case are still associated with the centroids (constant density) of the three neighboring triangles; i.e., $\bar{\mathcal{N}} = \bar{\mathcal{N}}(C)$ as before. Thus, the (possibly irregular) triangulations in [3] that can lead to second-order accuracy are characterized for each triangle $C$ as follows: the

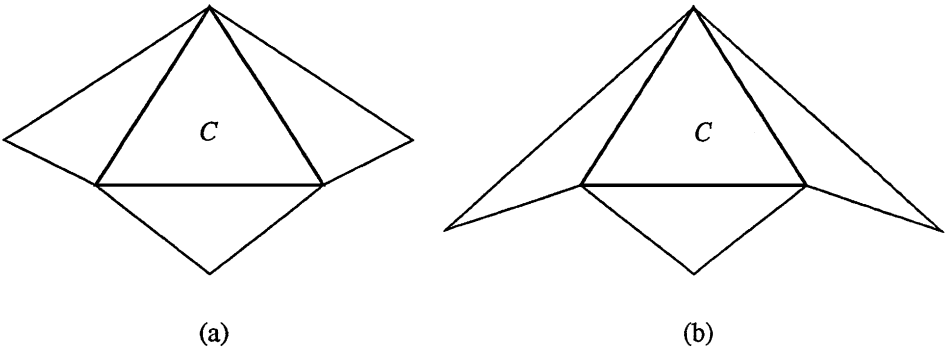


**FIG. 2.** Bad neighborhoods for Van Leer limiting at only the midpoints of $C$'s sides.

triangle determined by $C$'s three neighboring cells' centroids must contain the midpoints of $C$'s three sides. For example, this is violated in each configuration in Fig. 2—since, in each, the centroids of the left and the right triangle are both below the midpoints of $C$'s left and right side. Indeed, the function $L(x)$ given by the (signed) distance that the point $x$ is from the line containing the triangle $C$'s base is both linear and subject to Van Leer limiting at two of the three midpoints. (In fact, Fig. 2b is sufficiently distorted so that Van Leer limiting—over either $\mathcal{C}$ or *all* of $C$—limits one to the appropriate *constant* function, because the value of $L$ at $C$'s centroid lies outside the interval containing its values at the three neighboring centroids.) The result in all these cases is first-order accuracy locally for any limited approximation to $L$—and associated non-convergent gradient estimates.

In work that follows up on Durlofsky *et al.*, Liu [4, p. 704, Definition 3.1] uses the centroids of a neighborhood like Fig. 1g for $\bar{\mathcal{N}}$ instead. This now allows the exact approximation of linear functions at the three side-midpoints $\mathcal{C}$, since, now, $\mathcal{C} \subset C \subseteq \mathcal{H}(\bar{\mathcal{N}})$.

## ACKNOWLEDGMENTS

The author has had fruitful conversations with R. Lowrie, S. Mosso, and W. Rider. This work was supported by the U.S. Department of Energy. The publisher recognizes that the U.S. Government retains a non-exclusive royalty-free license to publish or reproduce the published form of this contribution, or allow others to do so, for U.S. Government purposes.

## REFERENCES

1. T. J. Barth and C. J. Jespersen, *The Design and Application of Upwind Schemes on Unstructured Meshes*, AIAA Paper 89-0366, 27th Aero. Sciences Meeting, January 1989.
2. J. K. Dukowicz and J. W. Kodis, Accurate conservative remapping (rezoning) for arbitrary Lagrangian–Eulerian computations, *SIAM J. Sci. Comput.* **8**, 305 (1987).
3. L. J. Durlofsky, B. Engquist, and S. Osher, Triangle based adaptive stencils for the solution of hyperbolic conservation laws, *J. Comput. Phys.* **98**, 64 (1992).
4. X.-D. Liu, A maximum principle satisfying modification of triangle based adaptive stencils for the solution of scalar hyperbolic conservation laws, *SIAM J. Numer. Anal.* **30**, 701 (1993).
5. B. Van Leer, Towards the ultimate conservative difference scheme. IV. A new approach to numerical convection, *J. Comput. Phys.* **23**, 276 (1977).
6. B. Van Leer, Towards the ultimate conservative difference scheme. V. A second-order sequel to Godunov's method, *J. Comput. Phys.* **32**, 101 (1979).